

Problem Set #7, Question 2

In this problem you were asked to derive the von Neumann stability criterion for the first upwind differencing method (FUDM). Following the derivation in Roache (1998) we will break this problem into two parts. First we will derive the stability criterion for FUDM diffusion-only and then we will derive FUDM advection-only. This approach is generally without difficulty, however, Roache does warn that *sometimes* this is not the case (see Kasahara, 1965).

Remembering that the partial differential equation for the advection-diffusion equation is:

$$\frac{\partial C}{\partial t} = -u \frac{\partial C}{\partial x} + \frac{\partial}{\partial x} \left(K_x \frac{\partial C}{\partial x} \right), \quad (\text{PS07_2-1})$$

in one-dimension. The advection-diffusion equation recast in the FUDM finite difference form is given by:

$$C_i^{n+1} = C_i^n - c (C_i^n - C_{i-1}^n) + d (C_{i+1}^n + C_{i-1}^n - 2C_i^n) \quad (\text{PS07_2-2})$$

where:

$$\begin{aligned} c &= \frac{u \Delta t}{\Delta x} \\ d &= \frac{K \Delta t}{\Delta x^2} \\ u &> 0 \end{aligned} \quad (\text{PS07_2-3})$$

FUDM Diffusion-only stability

This derivation is no different from the derivation we arrived at in the class notes for the forward time centered space (FTCS) finite difference scheme. But we will do it here to get warmed up.

First, consider the instabilities we observed in the FTCS example in the class web notes. They looked a lot like an oscilloscope trace, which immediately put us in a Fourier frame of mind, didn't it? However, in this particular case we will be doing our Fourier analysis in space rather than time. This does not present a problem as we can easily substitute wavenumber for frequency and then we can write, for one Fourier component,

$$\begin{aligned} C_i^n &= V_n e^{I i k_x \Delta x} \\ C_{i+1}^n &= V_n e^{I (i+1) k_x \Delta x} \\ C_{i-1}^n &= V_n e^{I (i-1) k_x \Delta x} \\ C_i^{n+1} &= V_{n+1} e^{I i k_x \Delta x} \end{aligned} \quad (\text{PS07_2-4})$$

where:

$$k_x \equiv \frac{2\pi}{\Lambda} \quad (\text{PS07_2-5})$$

is the wavenumber, the inverse of the wavelength (Λ). Here $I = \sqrt{-1}$, a small break with tradition because we use the i to represent different grid points.

Now if we multiply the wavenumber by the grid spacing we create an angle,

$$\theta \equiv k_x \Delta x \quad (\text{PS07_2-6})$$

If you think about it we think you can see this, a distance per some wavelength. Putting this θ back into the definitions of the Fourier component above we have,

$$\begin{aligned} C_i^n &= V_n e^{Ii\theta} \\ C_{i+1}^n &= V_n e^{I(i+1)\theta} \\ C_{i-1}^n &= V_n e^{I(i-1)\theta} \\ C_i^{n+1} &= V_{n+1} e^{Ii\theta} \end{aligned} \quad (\text{PS07_2-7})$$

As we say in the class notes on the web, you should get comfortable with these definitions because you will be using them for the rest of this problem (and in your future von Neumann stability analysis).

Substituting this Fourier component into the *diffusion-only* part of equation PS07_2-2 we get the following equation. Note: we will use just one Fourier component, but we could throw in the required summation signs and add to the complexity of this equation to include all of the Fourier components, but we wish to keep it simple for clarity's sake and, as you will see in a moment, it really isn't necessary.

$$V_{n+1} e^{Ii\theta} = V_n e^{Ii\theta} + d \left(V_n e^{I(i+1)\theta} + V_n e^{I(i-1)\theta} - 2V_n e^{Ii\theta} \right) \quad (\text{PS07_2-8})$$

If we group the V 's and divide through by $e^{Ii\theta}$, we get:

$$V_{n+1} = V_n \left[1 + d \left(e^{I\theta} + e^{-I\theta} - 2 \right) \right] \quad (\text{PS07_2-9})$$

We reduce this equation by applying the "*well known trigonometric identity*" ... you are uncertain as to what this identity is? Consider:

$$e^{I\theta} + e^{-I\theta} - 2 \equiv 2(\cos \theta - 1), \quad (\text{PS07_2-10})$$

if you don't believe us, we leave the derivation to you. However, applying this trigonometric identity and a little algebraic manipulation yields:

$$V_{n+1} = V_n [1 - 2d(1 - \cos \theta)], \quad (\text{PS07_2-11})$$

and the reason for not worrying about all the Fourier components becomes clear: $\cos \theta$ can only vary from negative one to positive one; for any Fourier component if one is stable, they all are stable.

Now, if we think of equation PS07_2-11 as a relationship between the value at step n and the next step $n + 1$ as an amplification relationship (amplify by a factor greater, less than or equal to one), then we can rewrite equation PS07_2-11 as:

$$V_{n+1} = GV_n \quad (\text{PS07_2-12})$$

and G can be written as:

$$G = \text{gain} = [1 - 2d(1 - \cos \theta)]. \quad (\text{PS07_2-13})$$

So long as G is less than or equal to one, that particular Fourier component can not “grow” and the numerical scheme will be stable. This is equivalent to writing:

$$|G|^2 \leq 1 \quad (\text{PS07_2-14})$$

or

$$-1 \leq 1 - 2d(1 - \cos \theta) \leq 1 \quad (\text{PS07_2-15})$$

Careful inspection of PS07_2-15 reveals that the right hand inequality will always be satisfied. You can prove this to yourself by considering the extreme values of the \cos function. When $\cos \theta$ is equal to -1, G will be equal to $1 - 4d$ and since d is always positive the right hand inequality will hold. When $\cos \theta$ equals +1, G will equal 1 and all's well. If we consider the left hand inequality, certainly it holds for $\cos \theta$ equal to +1 ($+1 \geq -1$). And when we consider $\cos \theta$ equal to -1, we have $-1 \leq 1 - 4d$ or:

$$d \leq \frac{1}{2}, \quad (\text{PS07_2-16})$$

which is the stability requirement for diffusion-only FUDM.

FUDM Advection-only stability

Now we shall address the stability requirements for FUDM in an advection-only sense. This is a bit of a departure from the way we handled the FTCS analysis in the notes. There we did the stability analysis for FTCS by putting the Fourier component(s) into the full FTCS finite difference equation. But as you no doubt recall, we demonstrated that for a purely advective system, FTCS was *unconditionally unstable* . . . which was considered not good. That led us to develop the FUDM and if we cannot demonstrate that the FUDM is stable for some choice of parameters for the purely advective case, what's the point?

So, substituting a Fourier component into the advection-only part of equation PS07_2-2 we obtain:

$$V_{n+1}e^{Ii\theta} = V_n e^{Ii\theta} - c(V_n e^{Ii\theta} - V_n e^{I(i-1)\theta}) \quad (\text{PS07_2-17})$$

for $c = \frac{u\Delta t}{\Delta x}$ and $u > 0$. Grouping the V 's and dividing through by $e^{Ii\theta}$ yields:

$$V_{n+1} = V_n [1 - c(1 - e^{-I\theta})] \quad (\text{PS07_2-18})$$

or

$$V_{n+1} = GV_n. \quad (\text{PS07_2-19})$$

We have been here before. All we need do is to require $|G|^2 \leq 1$. Ahhhhhh, but wait a minute, G is now complex. All we need to do to evaluate $|G|^2$ is to multiply the complex conjugates (*i.e.* $x - Iy$ is the complex conjugate of $x + Iy$). Having no fear, we press on:

$$\begin{aligned} |G|^2 &= \left| [1 - c + ce^{-I\theta}] \right|^2 \leq 1 \\ &= \left| [1 - c + c(\cos \theta - I \sin \theta)] \right|^2 \\ &= \left| [1 - c + c \cos \theta - Ic \sin \theta] \right|^2 \\ &= [(1 - c + c \cos \theta) - Ic \sin \theta] [(1 - c + c \cos \theta) + Ic \sin \theta] \\ &= (1 - c + c \cos \theta)^2 + RIc \sin \theta - RIc \sin \theta - I^2 c^2 \sin^2 \theta \\ &= (1 - c + c \cos \theta)^2 + c^2 \sin^2 \theta \leq 1 \end{aligned} \quad (\text{PS07_2-20})$$

where

$$R = (1 - c + c \cos \theta) \quad (\text{PS07_2-21})$$

representing a “real” number and making the equation compact enough to fit on the page. Now to play the game of evaluating the expression at the extreme values of $\cos \theta$ and $\sin \theta$ we need to expand PS07_2-20 out to individual terms ... unless you are comfortable evaluating expressions like $(1 - c + c \cos \theta)^2$ in your head. Actually it's not that difficult, but to be complete:

$$\begin{aligned} (1 - c + c \cos \theta)^2 + c^2 \sin^2 \theta &\leq 1 \\ 1 - 2c + c^2 + 2c \cos \theta(1 - c) + c^2(\cos^2 \theta + \sin^2 \theta) &\leq 1 \end{aligned} \quad (\text{PS07_2-22})$$

Here we have both $\cos \theta$ and $\sin \theta$ functions, but the extremes occur 90° out of phase with each other. So,

when $\theta = 0$:

$$\begin{aligned} 1 - 2c + c^2 + 2c(1 - c) + c^2(1 + 0) &\leq 1 \\ 1 - 2c + c^2 + 2c - 2c^2 + c^2 &\leq 1 \\ 1 &\leq 1 \end{aligned} \quad (\text{PS07_2-23})$$

which is true. When $\theta = \pi/2$:

$$\begin{aligned} 1 - 2c + c^2 + 2c(0)(1 - c) + c^2(0 + 1) &\leq 1 \\ 1 - 2c + 2c^2 &\leq 1 \\ -2c + 2c^2 &\leq 0 \\ 2c^2 &\leq 2c \\ c &\leq 1 \end{aligned} \quad (\text{PS07_2-24})$$

Which is what Roache got and what we were assuming in Lectures 15 and 16.

Now, we know that *someone* will ask, “what if $\theta = \pi/4$?” Well, let’s take a look. First of all, remember that:

$$\sin\left(\frac{\pi}{4}\right) = \sqrt{\frac{1}{2}} \quad \text{and} \quad \cos\left(\frac{\pi}{4}\right) = \sqrt{\frac{1}{2}} \quad (\text{PS07_2-25})$$

pushing onward then:

$$\begin{aligned} 1 - 2c + c^2 + 2c \left(\sqrt{\frac{1}{2}}\right) (1 - c) + c^2 \left[\left(\sqrt{\frac{1}{2}}\right)^2 + \left(\sqrt{\frac{1}{2}}\right)^2 \right] &\leq 1 \\ 1 - 2c + c^2 + \sqrt{2}c(1 - c) + c^2 &\leq 1 \\ 1 - 2c + c^2 + \sqrt{2}c - \sqrt{2}c^2 + c^2 &\leq 1 \\ c(-2 + \sqrt{2}) + c^2(2 - \sqrt{2}) &\leq 0 \quad (\text{PS07_2-26}) \\ -c(2 - \sqrt{2}) + c^2(2 - \sqrt{2}) &\leq 0 \\ c^2(2 - \sqrt{2}) &\leq c(2 - \sqrt{2}) \\ c^2 &\leq c \\ c &\leq 1 \end{aligned}$$

as before. And so we see that this stability requirement holds even for situations where θ has values in between the extreme values.

Since this was the FUDM, it is important to say a word about the direction of u . You may recall that the finite difference algorithm was slightly different if the flow was in the other direction (we have done the above analysis for $u > 0$). For $u < 0$ the FUDM algorithm would be:

$$C_i^{m+1} = C_i^m - c(C_{i+1}^m - C_i^m) \quad (\text{PS07_2-27})$$

We think that if you look at equation PS07_2-17, PS07_2-18, and PS07_2-20 you will see that in the end the complex conjugate will have the other sign and the results will be slightly different looking, but in the end, the same. Consider adding the Fourier component:

$$V_{n+1}e^{Ii\theta} = V_n e^{Ii\theta} - c(V_n e^{I(i+1)\theta} - V_n e^{Ii\theta}), \quad (\text{PS07_2-28})$$

grouping the V ’s and dividing through by $e^{Ii\theta}$ yields:

$$V_{n+1} = V_n [1 - c(e^{I\theta} - 1)] \quad (\text{PS07_2-29})$$

and now G looks like:

$$G = [1 - c(e^{I\theta} - 1)]. \quad (\text{PS07_2-30})$$

Requiring $|G|^2 \leq 1$ leads us to:

$$\begin{aligned}
 |G|^2 &= \left| [1 - c(e^{I\theta} - 1)] \right|^2 \\
 &= \left| [1 - c(\cos \theta + I \sin \theta - 1)] \right|^2 \\
 &= [1 + c - c \cos \theta + Ic \sin \theta] [1 + c - c \cos \theta - Ic \sin \theta] \quad (\text{PS07_2-31}) \\
 &= (1 + c - c \cos \theta)^2 + RIc \sin \theta - RIc \sin \theta + I^2 c^2 \sin^2 \theta \\
 &= (1 + c - c \cos \theta)^2 - c^2 \sin^2 \theta
 \end{aligned}$$

we skipped a few steps, but the derivation is essentially as before. Now without going through the expansion (it really wasn't necessary).

When $\theta = 0$

$$\begin{aligned}
 |G|^2 &= (1 + c - c(1)) - c^2(0) \\
 &= 1
 \end{aligned} \quad (\text{PS07_2-32})$$

which is true, $1 \leq 1$. When $\theta = \pi/2$:

$$\begin{aligned}
 &= (1 + c - c(0)) - c^2(1) \\
 &= (1 + c)^2 - c^2 \\
 &= 1 + 2c + c^2 - c^2 \\
 &= 1 + 2c \leq 1 \\
 c &\leq 0
 \end{aligned} \quad (\text{PS07_2-33})$$

which we already knew to be true, because $u < 0$ and so $c = \frac{u\Delta t}{\Delta x} < 0$. But this doesn't bound c yet. So let's consider:

$$|G|^2 \leq 1 \quad (\text{PS07_2-34})$$

is equivalent to:

$$-1 \leq G \leq 1 \quad (\text{PS07_2-35})$$

if we use:

$$\begin{aligned}
 -1 &\leq [1 - c(e^{I\theta} - 1)] \\
 &\leq 1 + c - c \cos \theta + Ic \sin \theta
 \end{aligned} \quad (\text{PS07_2-36})$$

When $\theta = 0$

$$\begin{aligned} -1 &\leq 1 + c - c(1)Ic(0) \\ -1 &\leq 1. \end{aligned} \tag{PS07_2-37}$$

When $\theta = \pi/2$:

$$\begin{aligned} -1 &\leq 1 + c - c(0) + Ic(1) \\ |-1|^2 &\leq |1 + c + Ic|^2 \\ 1 &\leq (1 + c + Ic)(1 + c - Ic) \\ &\leq (1 + c)^2 - I^2c^2 \\ &\leq (1 + c)^2 + c^2 \\ &\leq 1 + 2c + c^2 + c^2 \\ 1 &\leq 1 + 2c + 2c^2 \\ -2c &\leq 2c^2 \\ -1 &\leq c \end{aligned} \tag{PS07_2-38}$$

or $-1 \leq c \leq 0$. Since we know $c < 0$ this makes sense and if we were to multiply through by a -1 , we would get the same result we got before: $1 \geq c \geq 0$.

Now what about that pesky $\theta = \pi/4$? Well it's gritty, but consider:

$$\begin{aligned} -1 &\leq G \leq 1 && \tag{PS07_2-39} \\ -1 &\leq [1 - c(\cos \theta + I \sin \theta - 1)] \\ -1 &\leq 1 - c\frac{1}{\sqrt{2}} - Ic\frac{1}{\sqrt{2}} + c \\ |-1|^2 &\leq \left| 1 - c\left(\frac{1}{\sqrt{2}} - 1\right) - Ic\frac{1}{\sqrt{2}} \right|^2 \\ 1 &\leq \left[1 - c\left(\frac{1}{\sqrt{2}} - 1\right) - Ic\frac{1}{\sqrt{2}} \right] \left[1 - c\left(\frac{1}{\sqrt{2}} - 1\right) + Ic\frac{1}{\sqrt{2}} \right] \\ &\leq 1 - c\left(\frac{1}{\sqrt{2}} - 1\right) + Ic\frac{1}{\sqrt{2}} - c\left(\frac{1}{\sqrt{2}} - 1\right) + c^2\left(\frac{1}{\sqrt{2}} - 1\right)^2 - Ic^2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}} - 1\right) \dots \\ &\quad - Ic\left(\frac{1}{\sqrt{2}}\right) + Ic^2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}} - 1\right) - I^2c^2\left(\frac{1}{\sqrt{2}}\right)^2 \\ &\leq 1 - 2c\left(\frac{1}{\sqrt{2}} - 1\right) + c^2\left(\frac{1}{\sqrt{2}} - 1\right)^2 - I^2c^2\left(\frac{1}{\sqrt{2}}\right)^2 \\ &\leq 1 - \frac{2c}{\sqrt{2}} + 2c + c^2\left(\frac{1}{\sqrt{2}} - 1\right)^2 + \frac{c^2}{2} \end{aligned}$$

$$\begin{aligned}
&\leq 1 - \sqrt{2}c + 2c + c^2 \left(\frac{1}{2} - \frac{2}{\sqrt{2}} + 1 \right) + \frac{c^2}{2} \\
&\leq 1 - \sqrt{2}c + 2c + \frac{c^2}{2} - \frac{2c^2}{\sqrt{2}} + c^2 + \frac{c^2}{2} \\
&\leq 1 + c(2 - \sqrt{2}) + c^2(2 - \sqrt{2}) \\
-c(2 - \sqrt{2}) &\leq c^2(2 - \sqrt{2}) \\
-1 &\leq c
\end{aligned}$$

and we see that again we find the same result. For FUDM, then, when $u > 0$ stability requires that $0 \leq c \leq 1$ and when $u < 0$, $-1 \leq c \leq 0$.

References

- Kasahara, A., 1965, On certain finite difference methods for fluid dynamics, *U.S. Monthly Weather Review*, **93**(1), 27–31.
- Roache, P.J., 1998, *Fundamentals of Computational Fluid Dynamics*, Hermosa Publishers, Albuquerque, NM, 648 pg.